

OPERATOR SPACES AND RESIDUALLY FINITE-DIMENSIONAL C^* -ALGEBRAS

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ABSTRACT. For every operator space X the C^* -algebra containing it in a universal way is residually finite-dimensional (that is, has a separating family of finite-dimensional representations). In particular, the free C^* -algebra on any normed space so is. This is an extension of an earlier result by Goodearl and Menal, and our short proof is based on a criterion due to Exel and Loring.

1. INTRODUCTION

A C^* -algebra is called residually finite-dimensional (RFD) if it has a separating family of finite-dimensional representations. Clearly, every commutative C^* -algebra is such; as shown by Choi [5], the C^* group algebra on a free group with two generators is RFD. In 1990 Goodearl and Menal [12] proved that every C^* -algebra as an image of a residually finite-dimensional one; they have shown that the free C^* -algebra, $C^*\langle \Gamma \rangle$, on a set Γ of free generators subject to the norm restriction $\|x\| \leq 1$, $x \in \Gamma$, is residually finite-dimensional. The most recent development is due to Exel and Loring [8] who proved that residual finite-dimensionality is preserved by C^* -algebra coproducts.

The goal of this paper is to give an extension to the aforementioned result by Goodearl and Menal by invoking the concept of an operator space, coming from quantized functional analysis [6].

The construction of the free C^* -algebra on a set is the left adjoint functor to the forgetful functor sending a C^* -algebra A to the set of elements of its unit ball. One can consider, however, “less forgetful” functors, such as

$$A \mapsto \{\text{underlying normed space of } A\},$$

or still a more subtle one

$$A \mapsto \{\text{underlying operator space of } A\}$$

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In both cases there exist left adjoints, which we call the free C^* -algebra on a normed (resp. operator) space. The former construction is a particular case of the latter one. The Goodearl-Menal free C^* -algebra on a set Γ is then exactly the free C^* -algebra on the operator space of the form $\text{MAX}(l_1(\Gamma))$.

As the main result of this work, we show that the free C^* -algebra on any operator space is residually finite-dimensional. Also, we demonstrate freeness of some subalgebras of free C^* -algebras on operator spaces, discuss free commutative C^* -algebras on normed spaces, and conclude the paper with an open problem.

All C^* -algebras in this paper are assumed to be unital. However, all results have their immediate non-unital analogs, obtained by proceeding to unitizations.

2. PRELIMINARIES

A *matrix norm* on a vector space X is a collection of norms $\|\cdot\|_n, n \in \mathbb{N}$ on the vector spaces $M_n(X) = X \otimes M_n(\mathbb{C})$. Every C^* -algebra carries a natural matrix norm. Any linear map $f: X \rightarrow Y$ between two vector spaces gives rise to a natural map $f_{(n)}: M_n(X) \rightarrow M_n(Y)$. If X and Y carry matrix norms, then f is said to be *completely bounded* if $\|f\|_{cb} \stackrel{\text{def}}{=} \sup\{\|f_{(n)}\|_n: n \in \mathbb{N}\} < +\infty$, a *complete contraction* if $\|f\|_{cb} \leq 1$, and a *complete isometry* if every $f_{(n)}$ is an isometry. A vector space X together with a fixed matrix norm is called an (*abstract*) *operator space* if it is completely isometric to a subspace of the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space. (Equivalently: to a subspace of a C^* -algebra.) Operator spaces have been characterized by Ruan [18] as those spaces with matrix norms satisfying the three conditions:

- (i) $\|v \oplus \mathbf{0}_m\|_{n+m} = \|v\|_n$,
- (ii) $\max\{\|Bv\|_n, \|vB\|_n\} \leq \|B\| \|v\|_n$,
- (L^∞) $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$,

for all $v \in M_n(X)$, $w \in M_m(X)$, and $B \in M_n$; here $M_n = M_n(\mathbb{C})$, $\mathbf{0}_m$ is the null element of $M_m(X)$, and

$$v \oplus w = \begin{bmatrix} v & \mathbf{0} \\ \mathbf{0} & w \end{bmatrix}$$

If E is a normed space, then there is the maximal among all matrix norms on X making it into an operator space and such that $\|\cdot\|_1$ is the original norm on X . The resulting operator space is denoted by $\text{MAX}(E)$. The correspondence $E \mapsto \text{MAX}(E)$ is functorial, and for any bounded linear mapping $f: E \rightarrow F$ one has $\|f\|_{cb} = \|f\|$ [4, 7].

Theorem 2.1 (Noncommutative Hahn-Banach Theorem – Arveson [2], Wittstock [21]). *Let X be an operator space and Y be an operator subspace. Let $n \in \mathbb{N}$. Then every completely bounded linear mapping $f: Y \rightarrow M_n$ extends to a completely bounded mapping $\tilde{f}: X \rightarrow M_n$ in such a way that $\|f\|_{cb} = \|\tilde{f}\|_{cb}$. \square*

The operator spaces play in noncommutative analysis and geometry the same rôle as the normed spaces play in commutative analysis and geometry [6]. For an account of theory of operator spaces, including numerous examples, the reader is referred to [4, 6, 7, 18].

If A is a C^* -algebra, then $\text{Rep}(A, H)$ stands for the set of all (degenerate and non-degenerate) representations of A in H , that is, C^* -algebra morphisms $A \rightarrow \mathcal{B}(H)$, endowed with the coarsest topology, making all the mappings of the form

$$\text{Rep}(A, H) \ni \pi \mapsto \pi(x)(\xi) \in H, \quad x \in A, \quad \xi \in H$$

continuous. Clearly, this topology is inherited from $C_s(A, \mathcal{B}_s(H))$; here the subscript “ s ” stands for the topology of simple convergence, so that $\mathcal{B}_s(H)$ is the space $\mathcal{B}(H)$ endowed with the strong operator topology. The basic neighbourhoods of an element $\pi \in \text{Rep}(A, H)$ are of the form

$$\mathcal{O}_\pi[x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n; \epsilon]$$

$$\stackrel{\text{def}}{=} \{\eta \in \text{Rep}(A, H) : \|\pi(x_i)(\xi_i) - \eta(x_i)(\xi_i)\| < \epsilon, \quad i = 1, 2, \dots, n\}$$

This topology was considered by Exel and Loring [8], and it is finer than the Fell topology [9]. A representation $\pi \in \text{Rep}(A, H)$ is termed *finite-dimensional* if its essential space (the closure of $\pi(A)(H)$) is finite-dimensional. A C^* -algebra A is called *residually finite-dimensional*, or *RFD*, if it admits a family of finite-dimensional representations which separate points.

Theorem 2.2 (Exel and Loring [8]). *A C^* -algebra A is residually finite-dimensional if and only if the set of finite-dimensional representations is everywhere dense in $\text{Rep}(A, H)$. \square*

3. FREE C^* -ALGEBRAS ON OPERATOR SPACES

The following result seems to be well known in the C^* -algebra folklore.

Theorem 3.1. *Let X be a normed space. There exist a C^* -algebra, $C^*\langle X \rangle$, and an isometric embedding $i_X: X \hookrightarrow C^*\langle X \rangle$ such that $i_X(X)$ generates $C^*\langle X \rangle$ topologically, and for every C^* -algebra A and a contractive mapping $f: X \rightarrow A$ there exists a C^* -algebra morphism $\hat{f}: C^*\langle X \rangle \rightarrow A$ with $f = \hat{f} \circ i_X$. Such a pair $(C^*\langle X \rangle, i)$ is essentially unique. \square*

This result extends to the class of operator spaces, and the argument remains pretty standard. It follows the scheme of proving the existence and uniqueness of universal arrows in topological algebra and functional analysis, invented by Kakutani for free topological groups [14] and used extensively since then in various situations (cf. [1, 10, 11, 13, 15-17]).

Theorem 3.2. *Let X be an operator space. There exist a C^* -algebra, $C^*\langle X \rangle$, and a completely isometric embedding $i_X: X \hookrightarrow C^*\langle X \rangle$ such that $i_X(X)$ generates $C^*\langle X \rangle$ topologically, and for every C^* -algebra A and a completely contractive mapping $f: X \rightarrow A$ there exists a C^* -algebra morphism $\hat{f}: C^*\langle X \rangle \rightarrow A$ with $f = \hat{f} \circ i_X$. Such a pair $(C^*\langle X \rangle, i_X)$ is essentially unique.*

Proof. Denote by \mathfrak{F} the class of all pairs (A, j) where A is a C^* -algebra and $j: X \rightarrow A$ is a completely contractive linear mapping such that the image $j(X)$ topologically generates A . By identifying the isomorphic pairs, one can assume that \mathfrak{F} is a set. Let i_X stand for the diagonal product of mappings $\Delta\{j: (A, j) \in \mathfrak{F}\}$, viewed as a mapping from X to the C^* direct product $\mathfrak{A} = \prod_{(A, j) \in \mathfrak{F}} A$. Clearly, i_X is correctly

defined and it is a complete contraction. Denote by $C^*\langle X \rangle$ the C^* -subalgebra of the C^* -algebra \mathfrak{A} generated by the image $i_X(X)$. The universality and uniqueness of the pair $(C^*\langle X \rangle, i_X)$ are checked immediately. To prove that i_X is in fact a complete isometry, notice that by the very definition of an operator space, there is a pair $(A, j) \in \mathfrak{F}$ such that $j: X \rightarrow A$ is a complete isometry. \square

In category theoretic terms, the pair $(C^*\langle X \rangle, i)$ is the *universal arrow* [15] to the forgetful functor, \mathcal{F} , from the category of C^* -algebras and C^* -algebra morphisms to the category of operator spaces and complete contractions. The above result shows that \mathcal{F} has left adjoint, namely $C^*\langle \cdot \rangle$. This is another addition to the sphere of practical applicability of the Blackadar's general construction of a C^* -algebra in terms of generators and relations [3].

Theorem 3.2 is indeed an extension of theorem 3.1.

Proposition 3.3. *Let X be a normed space. Then the free C^* -algebra on X is canonically isomorphic to the free C^* -algebra on the maximal operator space $\text{MAX}(X)$ associated to X .*

Proof. Follows from the fact that the functor $X \mapsto \text{MAX}(X)$ is left adjoint to the forgetful functor from the category of operator spaces and complete contractions to the category of normed spaces and contractions. \square

Now we put the above construction in connection with the concept of the free C^* -algebra on a set Γ of free generators [3, 12].

Proposition 3.4. *Let Γ be a set. Then the free C^* -algebra on Γ is canonically isomorphic to the free C^* -algebra on the normed space $l_1(\Gamma)$ or, what is the same, the free C^* -algebra on the operator space $\text{MAX}(l_1(\Gamma))$.*

Proof. The functor of the form $\Gamma \mapsto l_1(\Gamma)$ is left adjoint to the functor sending a normed space to the set of elements of its unit ball. \square

4. REPRESENTATIONS OF FREE C^* -ALGEBRAS

The central result of our paper is the following.

Theorem 4.1. *The free C^* -algebra on an operator space, $C^*\langle X \rangle$, is residually finite-dimensional.*

Proof. Let $\pi \in \text{Rep}(C^*\langle X \rangle, H)$ for some Hilbert space H ; in view of the Exel-Loring criterion 2.2, it suffices to find a finite-dimensional representation in an arbitrary neighbourhood, U , of π . Assume that $U = \mathcal{O}_\pi[f_1, f_2, \dots, f_n; \xi_1, \xi_2, \dots, \xi_n; \epsilon]$ for some $n \in \mathbb{N}$, $f_1, f_2, \dots, f_n \in C^*\langle X \rangle$, $\xi_1, \xi_2, \dots, \xi_n \in H$, and $\epsilon > 0$.

Fix for each i a star-polynomial p_i in variables belonging to X with $\|f_i - p_i\| < \epsilon/2$. There exists a finite collection of elements $\{x_1, x_2, \dots, x_m\} \subset X \cup X^*$ such that each p_i is a polynomial in variables x_1, x_2, \dots, x_m . Denote by k an upper bound of the degrees of p_1, p_2, \dots, p_n relative to $\{x_1, x_2, \dots, x_m\}$. Let V be the subspace of H spanned by all elements of the form $\xi_1, \xi_2, \dots, \xi_n$ and $\pi(x_{i_1})\pi(x_{i_2}) \dots \pi(x_{i_l})(\xi_i)$, where $i_1, i_2, \dots, i_l \in \{1, 2, \dots, m\}$, $l \leq k$, $i = 1, 2, \dots, n$. Denote by p_V the orthogonal projection from H onto the finite-dimensional subspace V .

For every $x \in X$ let $\eta(x) = p_V \pi(x) p_V$. Since both the left and the right multiplication by an idempotent element in a C^* -algebra A are complete contractions with respect to the natural operator space structure on A , the linear mapping

$\eta: X \rightarrow \mathcal{B}(H)$ is completely contractive. By the universality of $C^*\langle X \rangle$, the mapping η lifts to a C^* -algebra morphism $\hat{\eta}: C^*\langle X \rangle \rightarrow \mathcal{B}(H)$, that is, $\hat{\eta} \in \text{Rep}(C^*\langle X \rangle, H)$. It follows from the definition of V that for every $i = 1, 2, \dots, n$ one has $\hat{\eta}(p_i)(\xi_i) = \pi(p_i)(\xi_i)$ and therefore for each $i = 1, 2, \dots, n$

$$\begin{aligned} & \| \pi(f_i)(\xi_i) - \hat{\eta}(f_i)(\xi_i) \| \leq \\ & \| \pi(f_i)(\xi_i) - \pi(p_i)(\xi_i) \| + \| \pi(p_i)(\xi_i) - \hat{\eta}(p_i)(\xi_i) \| + \| \hat{\eta}(p_i)(\xi_i) - \hat{\eta}(f_i)(\xi_i) \| < \\ & \epsilon/2 + 0 + \epsilon/2 = \epsilon, \end{aligned}$$

that is, $\hat{\eta} \in U$.

Finally, let q be any star-polynomial with variables from X . Since for any $x \in X$ one has $\hat{\eta}(x)(H) \subset V$ and $(\hat{\eta}(x))^*(H) \subset V$, one concludes that $\hat{\eta}(q)(H) \subset V$ as well. Since $\hat{\eta}$ is continuous and polynomials of the form q are dense in $C^*\langle X \rangle$, the essential space of $\hat{\eta}$ is V , that is, the representation $\hat{\eta}$ is finite-dimensional. \square

Corollary 4.2. *The free C^* -algebra over a normed space E , $C^*\langle E \rangle$, is residually finite-dimensional.* \square

This was announced (without a proof) in our survey [17].

Corollary 4.3 (Goodearl and Menal [12]). *The free C^* -algebra over any set Γ , $C^*\langle \Gamma \rangle$, is residually finite-dimensional.* \square

5. FREE C^* -SUBALGEBRAS

It is well known that if $i: X \rightarrow Y$ is a monomorphism, then its extension to universal objects, \hat{i} , need not be a monomorphism anymore. For example, the problem of describing homeomorphic embeddings of topological spaces, $i: X \rightarrow Y$, such that the monomorphism of the free topological groups which extends i is topological, proved to be fairly difficult and was solved only recently [19]. On the contrary, the similar problem for free locally convex spaces turned out to be readily amenable to the duality techniques [10, 11].

The noncommutative Hahn-Banach theorem and our Theorem 4.1 enable us to solve the similar problem for free C^* -algebras.

Theorem 5.1. *Let X be an operator space and let $Y \hookrightarrow X$ be an operator subspace. Then the C^* -subalgebra of $C^*\langle X \rangle$, generated by Y , is canonically isomorphic to the free C^* -algebra on Y .*

Proof. Denote by $i: Y \hookrightarrow X$ the complete isometry, and by $\hat{i}: C^*\langle Y \rangle \rightarrow C^*\langle X \rangle$ its extension to free C^* -algebras. Being residually finite-dimensional, the C^* -algebra $C^*\langle Y \rangle$ admits a monomorphism into the C^* -direct product of a family of finite-dimensional matrix algebras and therefore is isometric to a C^* -subalgebra of such a C^* -direct product. Now let $y \in C^*\langle Y \rangle$ and $\epsilon > 0$. By the above said, there exists an $n \in \mathbb{N}$ and a morphism $f: C^*\langle Y \rangle \rightarrow M_n$ with

$$\| f(y) \|_{M_n} > \| y \|_{C^*\langle Y \rangle} - \epsilon$$

By virtue of the noncommutative Hahn-Banach theorem 2.1, the restriction $f_Y = f|_{C^*\langle Y \rangle}$, which is clearly a complete contraction, extends to a complete contraction

$f_X: X \rightarrow M_n$. The latter mapping lifts to a C^* -algebra morphism $\tilde{f}: C^*\langle X \rangle \rightarrow M_n$, and the following diagram commutes:

$$\begin{array}{ccccc} Y & \xrightarrow{i_Y} & C^*\langle Y \rangle & \xrightarrow{f} & M_n \\ i \downarrow & & \tilde{i} \downarrow & & \parallel \\ X & \xrightarrow{i_X} & C^*\langle X \rangle & \xrightarrow{\tilde{f}} & M_n \end{array}$$

Now it is clear that

$$\left\| \tilde{f}(\tilde{i}(y)) \right\|_{M_n} = \|f(y)\|_{M_n} > \|y\|_{C^*\langle Y \rangle} - \epsilon,$$

which means, in view of arbitrariness of an $\epsilon > 0$, that $\|\tilde{i}(y)\|_{C^*\langle X \rangle} \geq \|y\|_{C^*\langle Y \rangle}$, and therefore $\tilde{i}: C^*\langle Y \rangle \rightarrow C^*\langle X \rangle$ is a C^* -algebra monomorphism, as desired. \square

6. THE COMMUTATIVE CASE

The universal C^* -algebra on an operator space is essentially non-commutative, and a sensible commutative analog of the construction exists for normed spaces only: the construction of the *free commutative C^* -algebra on a normed space E* , which we denote by $C_{com}^*\langle E \rangle$. It is the abelianization of the algebra $C^*\langle E \rangle$.

Lemma 6.1. *The spectrum of the free commutative C^* -algebra, $C_{com}^*\langle E \rangle$, on an infinite-dimensional normed space E is canonically homeomorphic to the closed unit ball $\text{BALL}(E'_\sigma)$ of the weak dual space of E .*

Proof. It is clear that the restriction map

$$\chi \mapsto \chi|_E: \Sigma(C_{com}^*\langle E \rangle) \rightarrow \text{BALL}(E'_\sigma)$$

is one-to-one and continuous. Since the spectrum $\Sigma(C_{com}^*\langle E \rangle)$ is compact, the mapping is a homeomorphism. \square

By Q the Hilbert cube is denoted. (Topologically, it is the countably infinite Tychonoff power $I^{\mathbb{N}_0}$ of the closed unit interval I .)

Theorem 6.2. *The free commutative C^* -algebra on any infinite-dimensional separable normed space E is isomorphic to $C(Q)$.*

Proof. According to Lemma 6.1, $C_{com}^*\langle E \rangle \cong C(\text{BALL}(E'_\sigma))$. Since the unit ball of the weak dual space to a separable normed space is homeomorphic to the Hilbert cube Q [20, Lemma 3], the statement follows. \square

7. CONCLUSION

The following may be of some relevance to quantized functional analysis.

Problem 7.1. Describe such pairs of operator spaces X, Y that their free C^* -algebras, $C^*\langle X \rangle$ and $C^*\langle Y \rangle$, are isomorphic. In particular, is it true or false that for any infinite dimensional separable normed space E the algebra $C^*\langle E \rangle$ is isomorphic to the free C^* -algebra on a countable set of free generators?

A similar problem of classifying topological spaces with isomorphic free topological groups had lead to an independent, if somewhat scholastic, problematics [1, 13, 17]

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